Critical state in type-II superconductors of arbitrary shape

G. P. Mikitik[1, 2] and E. H. Brandt[1]

[1] *Max-Planck-Institut für Metallforschung, D-70506 Stuttgart, Germany*
[2] *B. Verkin Institute for Low Temperature Physics & Engineering, National Ukrainian Academy of Sciences, Kharkov 61103, Ukraine*

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The well-known Bean critical state equations in general are not sufficient to describe the critical state of type-II superconductors when the sample shape is not symmetric. We show how one can find the critical state in superconductors of arbitrary shape. Analyzing a simple example of nonsymmetry, we demonstrate that in the general case, a perturbation of the current distribution in the critical state propagates into the sample smoothly in a diffusive way. This is in contrast to the usual Bean critical state where the current distribution changes abruptly at a narrow front.

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The concept of the critical state introduced by Charles Bean[1] is widely used to describe various physical phenomena in the vortex phase of type-II superconductors, see, e.g., Refs. 2,3 and citations therein. According to Bean, in the critical state of type-II superconductors with flux-line pinning, the driving force of the currents flowing in this state is balanced by the pinning force acting on the vortices. The critical state is characterized by the component of the current density flowing *perpendicular* to the flux lines, $j_{c\perp}$, since only this component generates a driving force. It is assumed in the critical-state theory that this $j_{c\perp}$ is known, i.e., it is a given function of the magnetic induction $\mathbf{B}$, $j_{c\perp} = j_{c\perp}(\mathbf{B})$, and the problem of this theory is to find the appropriate distribution of the magnetic fields and currents in the critical state. Below, to explain the physics with the least mathematical complications, we shall imply the simplest form: $j_{c\perp} = \text{const.}$, which is frequently used in practice, but an extension to the general case is straightforward. For simplicity, we also assume that the magnetic fields $\mathbf{H}$ in the superconductor considerably exceed the lower critical field H_{c1}, and so we may put $\mathbf{B} = \mu_0\mathbf{H}$.

If in the critical state a current-density component $j_\parallel$ *parallel* to the local magnetic field is also generated, the magnitude of $j_\parallel$ remains undefined, and one thus cannot in general find the distributions of the magnetic field $\mathbf{H}(\mathbf{r})$ and current density $\mathbf{j}(\mathbf{r})$ in the critical state. Indeed, to solve the Maxwell equations for $\mathbf{H}$,

$$\mathrm{rot}\mathbf{H} = \mathbf{j}, \quad \mathrm{div}\mathbf{H} = 0, \tag{1}$$

it is necessary to know the magnitude and direction of the currents $\mathbf{j}(\mathbf{r})$ in the sample.[4] However, one has only the *two* conditions:

$$j_\perp = j_{c\perp}, \quad \mathrm{div}\mathbf{j} = 0, \tag{2}$$

for *three* quantities: $j_\perp$, $j_\parallel$ and the angle defining the direction of $j_\perp$ in the plane normal to $\mathbf{H}(\mathbf{r})$. Thus, the existing critical-state theory based only on Eqs. (1), (2) is *not complete*.

The above Maxwell equations with conditions (2) can provide the description of the critical state when the shape of the superconductor is sufficiently symmetric and the external magnetic field is applied along a symmetry axis, so that some constraint on the directions of the currents is known in advance. For example, the direction of the currents is obvious for a slab in an external magnetic field parallel to its surface. For an infinitely long cylinder with arbitrary cross-section in a magnetic field parallel to its axis, the currents flow perpendicular to this axis, and the critical state problem is solved.[2] Another completely solvable case is infinitely thin flat superconductors,[3] for which the currents can flow only in the plane. However, in the case, e.g., of a thin rectangular platelet of *finite* thickness in a perpendicular magnetic field, the above critical state equations are already incomplete for determining the magnetic fields and currents in the critical state.[5,6]

We emphasize that even for *simple* experimental situations equations (1) together with conditions (2) can be insufficient for solving the critical-state problem. As an example that we shall analyze below, consider an infinite slab of thickness d. Let this slab fill the space $|x|$, $|y| < \infty$, $|z| \le d/2$, and be in a constant and uniform external magnetic field H_a ($H_a \gg J_c \equiv j_{c\perp}d$) directed along the z axis, i.e., perpendicularly to the slab plane. Let then a constant field h_{ax} ($J_c/2 \le h_{ax} \ll H_a$) be applied along the x axis, and after that the magnetic field h_{ay} ($h_{ay} \ll H_a$) is switched on in the y direction. This critical state problem is not fully defined. Indeed, the condition $\mathrm{div}\mathbf{j} = 0$ yields $j_z = 0$, i.e., the currents flow in the x-y planes. Then, to describe the critical state, we may use the parametrization:

$$\mathbf{j} = j_c(\varphi,\theta,\psi)(\cos\varphi(z),\ \sin\varphi(z), 0),$$
$$\mathbf{H}(z) = \mathbf{H}_a + \mathbf{h}(z),$$
$$\mathbf{h}(z) = (h_x(z), h_y(z), 0),$$

where $j_c(\varphi,\theta,\psi)$ is the magnitude of the critical current density when a flux-line element is given by the angles ψ and θ, $\tan\psi = h_y/h_x$, $\tan\theta = (h_x^2+h_y^2)^{1/2}/H_a$, while the current flows in the direction defined by the angle φ; all these angles generally depend on z. A dependence of j_c on the orientation of the local $\mathbf{H}$, $j_c(\varphi,\theta,\psi) = j_{c\perp}/[1 -$

$\cos^2(\varphi-\psi)\sin^2\theta]^{1/2}$,[6] appears if $\mathbf{j_c}$ is not perpendicular to this $\mathbf{H}$. However, at $H_a \gg h_{ax}$, h_{ay}, J_c, the field $\mathbf{H}$ is practically normal to the x-y planes where the currents flow ($\theta \approx 0$), and we may put $j_c(\varphi,\theta,\psi) = j_{c\perp}$. With this parametrization, the equation $\mathrm{div}\mathbf{H}=0$ is satisfied identically, while the Maxwell equation $\mathrm{rot}\mathbf{H}=\mathbf{j}$ reads

$$\frac{dh_x}{dz} = j_{c\perp}\sin\varphi, \qquad (3)$$

$$-\frac{dh_y}{dz} = j_{c\perp}\cos\varphi, \qquad (4)$$

and one has only *two* equations for the *three* functions $h_x(z)$, $h_y(z)$, $\varphi(z)$.

In real samples of nonsymmetric shape, adjacent flux lines may be slightly rotated relative to each other in the critical state. It is this rotation that generates a component of the current along the magnetic field. The rotation of flux-lines can lead to their mutual cutting.[2,7] Flux line cutting occurs when the component of the current density parallel to the magnetic field, $j_\parallel$, exceeds some longitudinal critical current density $j_{c\parallel}$. In this situation a vortex[8] or a vortex array[9] becomes unstable with respect to a helical distortion, and the growth of this distortion leads to flux-line cutting. When both $j_\parallel$ and $j_{c\perp}$ are equal to their critical values $j_{c\parallel}$ and $j_{c\perp}$, respectively, Clem's double critical state[7,10] occurs in the superconductor. In this case one has three conditions for the three quantities, and equations (1) are sufficient to describe the double critical state. However, in many real situations, $j_\parallel$ does *not* reach $j_{c\parallel}$, and flux cutting then does *not* occur in the critical state. It is such situations that we consider here. In particular, in the above example the projection of the current density on the magnetic field (which practically coincides with the z axis) is negligible, and flux cutting does not occur.

We now show how the critical state problem can be solved for superconductors of arbitrary shape. Let the critical state be known at some moment of time t, i.e., one has $\mathbf{H}=H(\mathbf{r})\boldsymbol{\nu}(\mathbf{r})$ inside the superconductor where the magnitude of the magnetic field, H, and the unit vector $\boldsymbol{\nu}$ are both known functions of the coordinates $\mathbf{r}$ at some external magnetic field $\mathbf{H}_a(t)$. The current density $\mathbf{j}(\mathbf{r})$ in the critical state follows from the Maxwell equation $\mathbf{j}=\mathrm{rot}\,(H(\mathbf{r})\boldsymbol{\nu}(\mathbf{r}))$, while the component of the current density perpendicular to the magnetic field is given by $\mathbf{j}_\perp = \mathbf{j}-\boldsymbol{\nu}(\boldsymbol{\nu}\mathbf{j}) \equiv j_{c\perp}\mathbf{n}_\perp(\mathbf{r})$. Here the last equality defines the unit vector $\mathbf{n}_\perp$. Let the external field infinitesimally (and slowly[11]) change by $\delta\mathbf{H}_a = \dot{\mathbf{H}}_a\delta t$. We now shall find the new critical state at the new external magnetic field $\mathbf{H}_a+\delta\mathbf{H}_a$.

Under the change of $\mathbf{H}_a$, the critical currents locally shift the vortices in the direction[12] of the Lorentz force $[\mathbf{j}\times\boldsymbol{\nu}]$; this shift generates an electric field directed along $[\boldsymbol{\nu}\times[\mathbf{j}\times\boldsymbol{\nu}]] = \mathbf{j}_\perp$, i.e., along the vector $\mathbf{n}_\perp$. Thus, we can represent the electric field $\mathbf{E}(\mathbf{r})$ in the form $\mathbf{E}=\mathbf{n}_\perp e$ where the scalar function $e(\mathbf{r})$ is the modulus of the electric field. Note that in contrast to the Bean assumption,[13] the electric field generally is not parallel to the total current density $\mathbf{j}(\mathbf{r})$. Using the Maxwell equation

$$\mathrm{rot}(e\mathbf{n}_\perp) = -\mu_0\dot{\mathbf{H}}, \qquad (5)$$

where $\dot{\mathbf{H}} \equiv \partial\mathbf{H}/\partial t$, and the equation

$$\mathrm{rot}\dot{\mathbf{H}} = \frac{\partial \mathbf{j}}{\partial t}, \qquad (6)$$

one can express the change of the magnetic fields and currents via *one scalar* function $e(\mathbf{r})$. This function can be found from the condition that in the critical state the absolute value of $\mathbf{j}_\perp$ is a given function of $\mathbf{B}$, $j_\perp = j_{c\perp}(\mathbf{B})$, or in the differential form,

$$\mathbf{j}_\perp\cdot\frac{\partial\mathbf{j}_\perp}{\partial t} = j_{c\perp}(\mathbf{B})\left(\frac{\partial j_{c\perp}(\mathbf{B})}{\partial\mathbf{B}}\cdot\mu_0\dot{\mathbf{H}}\right).$$

In our case when $j_{c\perp}$ =const., this condition reads

$$\mathbf{j}_\perp\cdot\frac{\partial\mathbf{j}_\perp}{\partial t} = 0.$$

Taking into account the definition of $\mathbf{j}_\perp$ and using the identities

$$H\dot{\boldsymbol{\nu}} = \dot{\mathbf{H}} - (\boldsymbol{\nu}\cdot\dot{\mathbf{H}})\boldsymbol{\nu},$$
$$H\dot{\boldsymbol{\nu}}\cdot\mathbf{j} = \dot{\mathbf{H}}\cdot\mathbf{j}_\perp,$$
$$\mathbf{j}\cdot\boldsymbol{\nu} = (\boldsymbol{\nu}\cdot\mathrm{rot}\boldsymbol{\nu})H,$$

we arrive at an equation for $e(\mathbf{r})$,

$$\mathbf{n}_\perp\cdot\{\mathrm{rot}\,\mathrm{rot}(e\mathbf{n}_\perp) - (\boldsymbol{\nu}\cdot\mathrm{rot}\boldsymbol{\nu})\,\mathrm{rot}(e\mathbf{n}_\perp)\} = 0. \qquad (7)$$

Continuity of the magnetic field on the surface of the superconductor, S, yields the boundary condition:

$$-\,\mathrm{rot}\,(e(\mathbf{r}_S)\mathbf{n}_\perp(\mathbf{r}_S)) = \mu_0\dot{\mathbf{H}}_a + \int\frac{[\mathbf{R}\times\mathrm{rot}\,\mathrm{rot}(e(\mathbf{r}')\mathbf{n}_\perp(\mathbf{r}'))]}{4\pi R^3}\,d\mathbf{r}', \qquad (8)$$

where $\mathbf{r}_S$ is a point on the surface S, $\mathbf{R}\equiv\mathbf{r}_S-\mathbf{r}'$, $R=|\mathbf{R}|$, and the integration is carried out over the volume of the sample. The right hand side of this boundary condition expresses $\mu_0\dot{\mathbf{H}}$ on the surface of the superconductor (but reaching from outside) with the use of the Biot-Savart law. If in the critical state of the superconductor there are also boundaries at which the direction of the critical currents changes discontinuously or which separate regions with $j_\perp = j_{c\perp}$ from regions with $j=0$,[14] the function $e(\mathbf{r})$ has to vanish at these boundaries. Otherwise, the electric field $e\mathbf{n}_\perp$ would be discontinuous there.

After determining the function $e(\mathbf{r})$, one can find the new critical state $\mathbf{H}(\mathbf{r})+\delta\mathbf{H}(\mathbf{r})$ using the definition $\delta\mathbf{H}(\mathbf{r}) = \dot{\mathbf{H}}\delta t$ and Eq. (5). We emphasized that the new critical state depends only on the previous state $\mathbf{H}(\mathbf{r})$ and on the change of the external field $\delta\mathbf{H}_a = \dot{\mathbf{H}}_a\delta t$. The dependence on $\delta\mathbf{H}_a$ follows from the proportionality

of e, $\dot{\mathbf{H}}$, $\partial \mathbf{j}_\perp/\partial t$ to $\dot{\mathbf{H}}_a$, which results from the linearity of Eqs. (5)-(8). Note that in agreement with the meaning of the critical state, the new state will be the same for different sweep rates of the external magnetic field, $\dot{\mathbf{H}}_a$, since it depends only on the *product* $\dot{\mathbf{H}}_a \delta t = \delta \mathbf{H}_a$. On the other hand, the electric field e plays an auxiliary role in the above description since it is proportional to $\dot{\mathbf{H}}_a$ rather than to $\delta \mathbf{H}_a$. The presented description also shows that the critical state generally depends on the history of its creation. In other words, it depends not only on the final value of the external magnetic field $\mathbf{H}_a$ but also on the sequence of steps $\delta \mathbf{H}_a$ that lead to this value.

The above approach, which is in essence the generalization of the appropriate analyses used for a slab,[10,15] can be summed up as follows: We add to the static equations (1) the quasistatic Maxwell equation (5). It is known[4] that for this set of the equations to be solvable, it has to be supplemented by some law $\mathbf{E}(\mathbf{j})$. We introduce this law from well-known physical ideas: At any given $\mathbf{j}$ and $\mathbf{B}$ (determined by the previous critical state), the *direction* of $\mathbf{E}$ follows from the formula $\mathbf{E} = [\mathbf{B} \times \mathbf{v}]$ where $\mathbf{v}$ is the vortex velocity caused by the Lorentz force $[\mathbf{j} \times \mathbf{B}]$. As to the *magnitude* of $\mathbf{E}$, it is found from the condition that $|\mathbf{j}_\perp| = j_{c\perp}$. In fact, this condition may be interpreted as a current–voltage law with $|\mathbf{E}| = e = 0$ at $j_\perp < j_{c\perp}$ and $e \to \infty$ at $j_\perp > j_{c\perp}$, which is usually implied in the description of the ideal critical state.[11]

Recently,[16,17] a variational principle was put forward to describe the critical state in superconductors. In deriving this principle Badía and López used Eqs. (1), (5) and a current–voltage law with $|\mathbf{E}| = 0$ at $j < j_c$ and $|\mathbf{E}| \to \infty$ at $j > j_c$. However, the physical idea on the *direction* of the electric field was not incorporated in their theory. For some situations this leads to contradiction with existing concepts.[8,9] In particular, in their so-called isotropic model with H-independent j_c, the electric field $\mathbf{E}$ is parallel to $\mathbf{j}$, and hence a nonzero $\mathbf{E}$ along $\mathbf{H}$ appears even for an infinitesimal longitudinal component of $\mathbf{j}$, i.e., flux-cutting occurs without any threshold $j_{c\|}$.

To illustrate the obtained results, we now consider the example mentioned above. In the case of the slab, equation (7) for the electric field e takes the form:

$$e'' - (\varphi')^2 e = 0, \tag{9}$$

where $\varphi' \equiv \partial\varphi/\partial z$ and $e'' \equiv \partial^2 e/\partial z^2$. For the angle φ we obtain from Eqs. (5) and (6):

$$\mu_0 j_{c\perp} \frac{\partial \varphi}{\partial t} = 2e'\varphi' + e\varphi''. \tag{10}$$

These equations complement Eqs. (3), (4), and now we have *four equations for four functions*. The boundary conditions to Eqs. (3), (4), (9), (10) at $z = d/2$ are

$$h_x = h_{ax}, \quad h_y = h_{ay}(t), \tag{11}$$

$$(e\cos\varphi)' = -\mu_0 \frac{dh_{ay}(t)}{dt}, \quad (e\sin\varphi)' = 0, \tag{12}$$

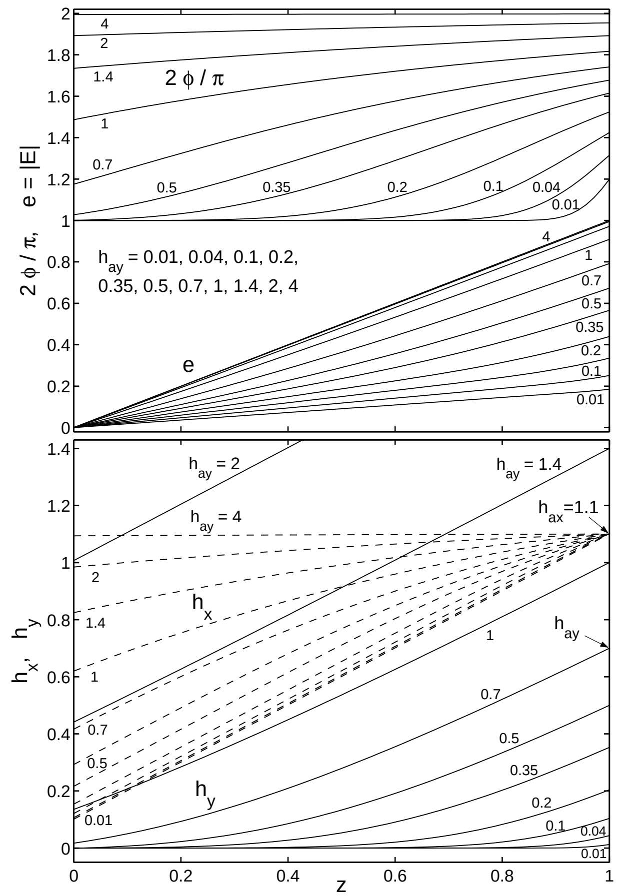

FIG. 1: Profiles of the angle of currents $\varphi(z)$, magnitude of electric field $e(z)$ (top) and magnetic field components $h_x(z)$ (dashed lines), $h_y(z)$ (solid lines) (bottom) in the critical states of the slab described by Eqs. (3), (4), (9), (10), (11) - (15); $h_{ax} = 1.1$ and $h_{ay} = 0.01$, 0.04, 0.1, 0.2, 0.35, 0.5, 0.7, 1, 1.4, 2, 4. We start at $h_{ay} = 0$ with $h_x(z) = h_{ax} - 1 + z$, $h_y(z) = 0$, and $\phi(z) = \pi/2$. Here z is in units $d/2$, h_x and h_y in units $j_{c\perp}d/2 = J_c/2$, and e in units $\mu_0(dh_{ay}/dt)d/2$.

or equivalently, conditions (12), which follow from formula (8), can be rewritten in the form:

$$e' = -\mu_0 \frac{dh_{ay}(t)}{dt}\cos\varphi, \quad e\varphi' = \mu_0 \frac{dh_{ay}(t)}{dt}\sin\varphi. \tag{13}$$

Taking into account the symmetry of the problem,[18] it is sufficient to solve equations (3), (4), (9), (10) in the region $0 \le z \le d/2$. At $z = 0$, where the direction of the currents changes discontinuously, one has the additional condition for e,

$$e(0) = 0. \tag{14}$$

Since after switching on h_{ax}, the critical currents flow in the y direction, we have the following initial state for Eq. (10):

$$\varphi(z, t = 0) = \pi/2, \tag{15}$$

where the moment $t = 0$ corresponds to the beginning of switching on h_{ay}.

In agreement with the general considerations given above, it follows from Eq. (10) and $e \propto dh_{ay}/dt$ that the angle φ is a function of z and h_{ay} rather than of z and t. In fact, this equation describes $\varphi(z)$ in the sequence of the critical states developed in the process of increasing h_{ay}. The solution of equations (3), (4), (9), (10) with conditions (11) - (15) is shown in Fig. 1. Interestingly, when h_{ay} increases, the *other component* of the magnetic field, h_x, penetrates further into the slab [at $h_{ay} \sim J_c \equiv j_{c\perp} d$, $h_x(z)$ almost coincides with h_{ax}], and the angle φ tends to π. In other words, with increasing h_{ay} the initial critical state for the component h_x relaxes, while the critical state for h_y is developed. Note that we should arrive at a different critical state with $\varphi = \pi/2 + \arctan(h_{ay}/h_{ax})$ if the x and y components of the external field were increased simultaneously [$h_{ay}(t)/h_{ax}(t)$ =const.] from zero to the same values h_{ax}, $h_{ay} \sim J_c$. Thus, the dependence of the critical state on its prehistory is clearly seen even in this simple example.

Figure 1 also reveals the following two interesting features of the critical state: (a) The visible penetrating front of h_y reaches the center of the slab when h_{ay} is still less than $J_c/2$, the field of full penetration in the Bean case. (b) The change of the angle $\varphi(z, h_{ay})$ has *diffusive character*. This is in stark contrast to the usual Bean critical state, in which any change of the current direction occurs inside a narrow front.

Interestingly, equations (3), (4), (9), (10) are applicable also to a number of other physical problems if the boundary and initial conditions are changed appropriately. In particular, these equations also describe the usual Bean critical state in the slab, corresponding to a *discontinuous* solution $\varphi(z)$. Using these equations, one can also investigate the low-frequency response of the slab to a *circularly* polarized ac field applied in the plane of the sample perpendicularly to the large magnetic field H_a.[19] It is clear from the data of Fig. 1 that this response will differ from the response to a linearly polarized ac field, for which the analysis based on the usual Bean model is applicable. These equations also enable one to consider the vortex-shaking effect:[20] If the field H_a is not uniform in the plane of the slab, and thus a sheet current $\mathbf{J}$ flows in it, a small ac field applied along the current leads to a continuous drift of vortices in the direction $[\mathbf{J} \times \mathbf{H}_a]$. It turns out that Eqs. (3), (4), (9), (10) have a solution that reproduces this result of Ref. 20, obtained there by a different method using geometrical arguments.

In summary, we have extended the critical-state theory to the general case when the sample is not sufficiently symmetric, or when the external field is not along a symmetry axis or its direction changes in some complex manner. In such situations the currents in the critical state need not be perpendicular to the local magnetic fields, and a *longitudinal component* of the currents with respect to these fields exists in the superconductor. When the magnitude of the longitudinal current density $j_\parallel$ does not exceed some critical value $j_{c\parallel}(\mathbf{B})$, the critical state can be found using our Eqs. (5) - (8). Such a state, in general, essentially differs from the usual Bean critical state. In other words, the Bean state is only a *special case* of the general critical state. When the component $j_\parallel$ reaches $j_{c\parallel}$ in the sample (or in part of its volume), Clem's double critical state develops there.[7,10]

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[12] In the case of anisotropic pinning the direction of the shift may differ from $[\mathbf{j} \times \boldsymbol{\nu}]$, see Ref. 6. But it is important here that even in this case, the direction of the shift can be expressed[6] via the directions of $\mathbf{j}$ and $\boldsymbol{\nu}$. Below, to avoid mathematical complications, we consider the simplest case.

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